ANALYTICAL GEOMETRY IN THREE DIMENSIONS OR SOLID GEOMETRY

Co-ordinates of a point in 3-space

Consider three mutually perpendicular lines $OX, OY$ & $OZ$ in 3-space with $O$ as origin. Let $P$ be a point in 3-space. Draw $PQ \perp$ to the plane $XOY$.

Draw $QA$ & $QB$ parallel to $OY$ & $OX$ to meet $OX$ at $A$ & $OY$ at $B$.

Let $OA = x, OB = y \& QP = z$

Then $(x, y, z)$ are taken as coordinates the point $P$. Three mutually $\perp$ lines $OX, OY$ & $OZ$ divide 3-space into eight equal parts called Octants. Co-ordinates of any point in the plane $XOY$ will be of the form $(x, y, o)$, in $XOZ$ plane $(x, o, z)$ in $YOZ$ plane $(o, y, z)$ co-ordinates of any point on $x$-axis are $(x, o, o)$ on $y$-axis, $(o, y, o)$ and $z$-axis $(o, o, z)$. Co-ordinates of origin $(o, o, o)$. The position vector of $P$ is $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ is also denoted by $\mathbf{r}$.

Distance between two points

$P(x_1, y_1, z_1) \& Q(x_2, y_2, z_2)$

using vectors, $\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} \& \overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\therefore \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$\therefore |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$\therefore$ Distance between two points $(x_1, y_1, z_1) \& Q(x_2, y_2, z_2)$ is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Section Formula

Let $R$ divide $PQ$ in the ratio $m : n$

then $\frac{PR}{RQ} = \frac{m}{n}$

ie $nPR = mRQ$

$\therefore n\overrightarrow{PR} = m\overrightarrow{RQ}$

ie $n(\overrightarrow{OR} - \overrightarrow{OP}) = m(\overrightarrow{OQ} - \overrightarrow{OR})$

ie $n\overrightarrow{OR} - n\overrightarrow{OP} = m\overrightarrow{OQ} - m\overrightarrow{OR}$

$\therefore (m + n)\overrightarrow{OR} = m\overrightarrow{OQ} + n\overrightarrow{OP}$
\[ \overrightarrow{OR} = \frac{m\overrightarrow{OQ} + n\overrightarrow{OP}}{m+n} \]

\( OR \) represents the position vector of \( R \) which divides \( P & Q \) in the ratio \( m : n \).

If \( R \) is the mid-point, then \( \overrightarrow{OR} = \frac{\overrightarrow{OQ} + \overrightarrow{OP}}{2} \)

Cartesian co-ordinates of \( R \) are given by \( \left( \frac{mx_1 + nx_1}{m+n}, \frac{my_1 + ny_1}{m+n}, \frac{mz_1 + nz_1}{m+n} \right) \)

if \( R \) lie out segment \( PQ \). Then co-ordinates of \( R \) are \( \left( \frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n} \right) \)

**Direction cosines of a line**

Let \( P \) be any point in plane, let the line \( OP \) make angles \( \alpha, \beta, \gamma \) with co-ordinate axes then \( \cos \alpha, \cos \beta, \cos \gamma \) are defined as directions cosines and usually denoted as \( l, m, n \).

To prove that \( l^2 + m^2 + n^2 = 1 \). Draw \( PB \perp OY \), then \( \cos \beta = \frac{y}{r} \) where \( OP = r \).

\[ \therefore y = r \cos \beta = km \]

Similarly by dropping \( \perp \) to \( OX \& OZ \), we can show that \( x = r \cos \alpha = kr \) & \( z = r \cos \gamma = nr \)

Squaring and adding, we get
\[ x^2 + y^2 + z^2 = l^2r^2 + m^2r^2 + n^2r^2 = (l^2 + m^2 + n^2)r^2 \]

but \( x^2 + y^2 + z^2 = r^2 \)
\[ \therefore l^2 + m^2 + n^2 = 1 \]

**Direction ratios of a line**

If the direction cosines of a line are proportional to \( a, b, c \) then \( a, b, c \) are called Direction Ratios of the line.

ie \( \frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k \) (say)

then \( l = ak \), \( m = bk \), \( n = ck \)
\[ \therefore l^2 + m^2 + n^2 = a^2k^2 + b^2k^2 + c^2k^2 \]

but \( l^2 + m^2 + n^2 = 1 \)
\[ \therefore (a^2 + b^2 + c^2)k^2 = 1 \]

ie \( k^2 = \frac{1}{a^2 + b^2 + c^2} \) \( \therefore k = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \)
If \(a, b, c\) are the direction ratios of any line then direction cosines are
\[
\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \quad \frac{c}{\sqrt{a^2 + b^2 + c^2}}
\]

If \(P(x_1, y_1, z_1)\) and \(Q(x_2, y_2, z_2)\) are any two points in 3-space. The direction ratios of \(PQ\) are given by \(x_2 - x_1, y_2 - y_1, z_2 - z_1\) and hence the direction cosines are
\[
\frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}, \quad \frac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}, \quad \frac{z_2 - z_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}
\]

**Note:** Co-ordinates of a unit vector represents the direction cosines of the line of the vector. For any other vectors the co-ordinates represent the direction ratios.

**Examples**

1. Find the distance between the points \((4, 3, -6)\) \& \((-2, 1, -3)\).

Solution: Let \(P = (4, 3, -6), Q = (-2, 1, -3)\)
\[
PQ = \sqrt{(4 + 2)^2 + (3 - 1)^2 + (-6 + 3)^2} = \sqrt{6^2 + 2^2 + (-3)^2} = \sqrt{36 + 4 + 9} = 7
\]

2. Show that the points \((-2, 3, 5), (1, 2, 3) \& (7, 0, -1)\) are colinear.

Solution: Let \(A = (-2, 3, 5), B = (1, 2, 3), C = (7, 0, -1)\)
\[
AB = \sqrt{(-2 - 1)^2 + (3 - 2)^2 + (5 - 3)^2} = \sqrt{9 + 1 + 4} = \sqrt{14}
\]
\[
BC = \sqrt{(1 - 7)^2 + (2 - 0)^2 + (3 + 1)^2} = \sqrt{36 + 4 + 16} = \sqrt{56} = 2\sqrt{14}
\]
\[
AC = \sqrt{(-2 - 7)^2 + (3 - 0)^2 + (5 + 1)^2} = \sqrt{81 + 9 + 36} = \sqrt{126} = 3\sqrt{14}
\]
\[
AB + BC = \sqrt{14} + 2\sqrt{14} = 3\sqrt{14} = AC
\]
\[
\therefore \text{The points } A, B, C \text{ are colinear.}
\]

**Alternate Method**

Direction ratios of \(AB\) are \(-2 - 1, 3 - 2, 5 - 3\) ie \(-3, 1, 2\)

Direction ratios of \(BC\) are \(-1 - 7, 2 - 0, 3 + 1\) ie \(-6, 2, 4\) ie \(-3, 1, 2\)

Direction ratios of \(AB \& BC\) are same. \(
\therefore \text{A, B \& C are colinear.}
\)

3. Show that the points \((3, 2, 2), (-1, 1, 3), (0, 5, 6), (2, 1, 2)\) lie on a sphere whose centre is \((1, 3, 4)\). Find also the radius of the sphere.

Solution: Let the given points be \(P = (3, 2, 2), Q = (-1, 1, 3), R = (0, 5, 6) \& S = (2, 1, 2)\). Let \(C = (1, 3, 4)\) be the centre.
\[
CP = \sqrt{(1 - 3)^2 + (3 - 2)^2 + (4 - 2)^2} = \sqrt{4 + 1 + 4} = 3
\]
\[
CQ = \sqrt{(1 + 1)^2 + (3 - 1)^2 + (4 - 3)^2} = \sqrt{4 + 1 + 1} = 3
\]
\[
CR = \sqrt{(0 - 1)^2 + (5 - 3)^2 + (6 - 4)^2} = \sqrt{1 + 4 + 4} = 3
\]
\[
CS = \sqrt{(2 - 1)^2 + (1 - 3)^2 + (2 - 4)^2} = \sqrt{1 + 4 + 4} = 3
\]


4. Find the co-ordinates of the point which divide the line joining the points $(2, -4, 3)$ & $(-4, 5, -6)$ in the ratio $2 : 1$.

Solution: Let $\vec{a} = (2, -4, 3)$, $\vec{b} = (-4, 5, -6)$

Point of section is given by $\frac{2\vec{b} + \vec{a}}{2+1}$

$$= \frac{2(-4, 5, -6) + 1(2, -4, 3)}{2+1}$$

$$= \frac{(-8 + 2, 10 - 4, -12 + 3)}{3}$$

$$= (-6, 6, -9)$$  

$$= (-2, 2, -3)$$

5. Find the co-ordinates of the point which divide the line joining $(3, 2, 1)$ & $(1, 3, 2)$ in the ratio $-2 : 1$.

Solution: Let $\vec{a} = (3, 2, 1)$, $\vec{b} = (1, 3, 2)$

Point of section is given by $\frac{-2\vec{b} + \vec{a}}{-2+1}$

$$= \frac{-2(1, 3, 2) + 1(3, 2, 1)}{-1}$$

$$= \frac{(-2, -6, -4) + (3, 2, 1)}{-1}$$

$$= (1, -4, -3)$$  

$$= (-1, 4, 3)$$

6. Find the ratios in which $XY$ plane divides the join of $(-3, 4, -8)$ & $(5, -6, 4)$. Also obtain the coordinates of the point of section.

Solution: Let $\vec{a} = (-3, 4, -8)$, $\vec{b} = (5, -6, 4)$

Let $k : 1$ be the ratio.

Point of section is given by $\frac{k\vec{b} + \vec{a}}{k+1} = \left(\frac{5k-3}{k+1}, \frac{-6k+4}{k+1}, \frac{4k-8}{k+1}\right)$

This point lies on $XY$ plane

$$\frac{4k-8}{k+1} = 0 \Rightarrow 4k = 8 \Rightarrow k = 2$$

$\therefore$ ratio is $2 : 1$ and co-ordinates of the point are $\left(\frac{7}{3}, \frac{-8}{3}, 0\right)$

7. Direction ratios of a line are $6, 2, 3$. Find the direction cosines.

Solution: Now $\sqrt{6^2 + 2^2 + 3^2} = \sqrt{36+4+9} = 7$

$\therefore$ direction cosines are $\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$
8. Find the direction cosines of the line joining the points (1, 4, – 3) & (4, 7, – 6).

Solution : Direction ratios of the line joining the points (1, 4, – 3) & (4, 7, – 6) are 4 – 1, 7 – 4, – 6 + 3

ie 3, 3, – 3.

:\[ \text{direction cosines are } \frac{3}{\sqrt{9+9+9}}, \frac{3}{\sqrt{9+9+9}}, \frac{-3}{\sqrt{9+9+9}} \]

ie \[ \frac{3}{3\sqrt{3}}, \frac{3}{3\sqrt{3}}, \frac{-3}{3\sqrt{3}} \]

To find the angle between two lines in 3-space

Let \( OA \) & \( OB \) be two lines whose direction cosines are \( l_1, m_1, n_1 \) & \( l_2, m_2, n_2 \).

\[
\begin{align*}
\vec{a} &= \vec{OA} = l_1\hat{i} + m_1\hat{j} + n_1\hat{k}, \quad \vec{b} = \vec{OB} = l_2\hat{i} + m_2\hat{j} + n_2\hat{k} \\
\vec{a} \cdot \vec{b} &= l_1l_2 + m_1m_2 + n_1n_2 \\
\text{but } \vec{a} \cdot \vec{b} &= ab\cos\theta \\
\therefore \cos\theta &= \frac{\vec{a} \cdot \vec{b}}{ab} = \frac{l_1l_2 + m_1m_2 + n_1n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2} \sqrt{l_2^2 + m_2^2 + n_2^2}} \\
&= \frac{l_1l_2 + m_1m_2 + n_1n_2}{l_1l_2 + m_1m_2 + n_1n_2}
\]

If \( \theta = \frac{\pi}{2} \), \( \cos \frac{\pi}{2} = 0 \) \( \therefore l_1l_2 + m_1m_2 + n_1n_2 = 0 \)

\( \therefore \) condition for two lines to be \( \perp \) is \( l_1l_2 + m_1m_2 + n_1n_2 = 0 \)

If \( a_1, b_1, c_1 \) & \( a_2, b_2, c_2 \) are direction ratios of two lines. Then angle \( \theta \) between them is given by

\[
\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}
\]

lines will be perpendicular if \( a_1a_2 + b_1b_2 + c_1c_2 = 0 \).

Note :- Two lines will be parallel if direction cosines of the two lines are same or if the direction ratios are proportional.

The Plane

To find the equation of the plane passing through the point \((x_1, y_1, z_1)\) and having \( a, b, c \) as the direction ratios of the normal to the plane.

Let \( A(x_1, y_1, z_1) \) be a point in the plane.

Let \( P(x, y, z) \) be any point in the plane then direction ratios of \( AP \) are given by \( x-x_1, y-y_1, z-z_1 \) as this is perpendicular to the normal to the plane, we have

\[
a(x-x_1) + b(y-y_1) + c(z-z_1) = 0
\]

ie \( ax + by + cz - (ax_1 + by_1 + cz_1) = 0 \)

which is a first degree equation in \( x, y \) & \( z \).
Note: -

1. \( z = 0 \) represents the equation to the \( XOY \) plane.
2. \( y = 0 \) represents the equation of the \( ZOX \) plane.
3. \( x = 0 \) represents the equation of \( YOZ \) plane.

To prove that \( ax + by + cz + d = 0 \) a first degree equation in \( x, y, z \) represents a plane.

Let \( P(x_1, y_1, z_1) \) & \( Q(x_2, y_2, z_2) \) be any two points satisfying \( ax + by + cz + d = 0 \).

Then
\[
ax_1 + by_1 + cz_1 + d = 0 \tag{1}
\]
\[
ax_2 + by_2 + cz_2 + d = 0 \tag{2}
\]

Multiply (2) by \( k \) & add to (1)
\[
k(ax_2 + by_2 + cz_2 + d) + (ax_1 + by_1 + cz_1 + d) = 0
\]

\[ax_2 + by_2 + cz_2 + d = \frac{k(x_2 + x_1)}{k + 1}, \quad \frac{ky_2 + y_1}{k + 1}, \quad \frac{kz_2 + z_1}{k + 1} + \frac{d}{k + 1} = 0
\]

This shows that the point whose coordinates are \( \left( \frac{kx_2 + x_1}{k + 1}, \frac{ky_2 + y_1}{k + 1}, \frac{kz_2 + z_1}{k + 1} \right) \) satisfy the equation \( ax + by + cz + d = 0 \).

Thus every point on the line joining \( P \) & \( Q \) lie on the locus. \( \therefore \) The equation represents the plane.

To find the length of the perpendicular from \( (x_1, y_1, z_1) \) upon the plane \( ax + by + cz + d = 0 \).

Let \( P(x_1, y_1, z_1) \) be the given point & \( PA \) is the length of the perpendicular on the plane \( ax + by + cz + d = 0 \).

Let \( Q(x, y, z) \) be any point in the plane, \( AP \) is the projection of \( QP \) upon the normal to the plane if \( \hat{n} \) is the unit normal vector of the plane then
\[
\hat{n} = \frac{a\hat{i} + b\hat{j} + c\hat{k}}{\sqrt{a^2 + b^2 + c^2}}
\]
\[
AP = \overrightarrow{QP} \cdot \hat{n} = \left( (x_1 - x)\hat{i} + (y_1 - y)\hat{j} + (z_1 - z)\hat{k} \right) \cdot \hat{n}
\]
\[
= \frac{a(x_1 - x) + b(y_1 - y) + c(z_1 - z)}{\sqrt{a^2 + b^2 + c^2}}
\]
\[
= \frac{ax_1 + by_1 + cz_1 - (ax + by + cz)}{\sqrt{a^2 + b^2 + c^2}}
\]
\[
= \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}
\]
\[
\therefore \ ax + by + cz = -d
\]
\[
\therefore \text{ length of the perpendicular } \ AP = \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}
\]

Note: -

Two points \( (x_1, y_1, z_1) \) & \( (x_2, y_2, z_2) \) lie on the same side or on opposite sides of the plane \( ax + by + cz + d = 0 \) according as \( ax_1 + by_1 + cz_1 + d \) & \( ax_2 + by_2 + cz_2 + d \) are of same sign or of opposite signs.
Normal form of the equation of the plane

Let $p$ represents length of the $\perp$ from the origin upon the plane $ax + by + cz + d = 0$

then $p = \frac{d}{\sqrt{a^2 + b^2 + c^2}}$

$Ix + my + nz = p$ where $I, m, n$ are the direction cosines of the normal to the plane, is taken as the normal equation of the plane.

To find the angle between two planes

$a_1x + b_1y + c_1z + d_1 = 0$ \& $a_2x + b_2y + c_2z + d_2 = 0$

The two planes intersect along a line, the angle between two planes is nothing but angle between two normals of the plane. \( \therefore \) if $\theta$ is the angle between two planes

then $\cos \theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}$ \& planes will be $\perp$ if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Intercept form of the equation if the plane

Let $a, b \& c$ be the intercept made by the plane on the co-ordinate axes, ie the plane passes through the points $(a, 0, 0)$, $(0, b, 0)$ \& $(0, 0, c)$

Let the equation to the plane be $ax + by + cz + d = 0$.

$(a, 0, 0)$ lies on the plane $\therefore a + d = 0 \therefore a = -\frac{d}{a}$

$(0, b, 0)$ lies on the plane $\therefore b + d = 0 \Rightarrow b = -\frac{d}{b}$

$(0, 0, c)$ lies on the plane $\therefore c + d = 0 \Rightarrow c = -\frac{d}{c}$

$\therefore$ equation of the plane is $-\frac{d}{a} x - \frac{d}{b} y - \frac{d}{c} z + d = 0$

ie $\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$

ie $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + 1$ is the equation of the plane in the intercept form.

Examples

(1) Find the equation of the plane passing through the point $(1, 2, 3)$ and having the vector $\hat{n} = 2\hat{i} - \hat{j} + 3\hat{k}$ as normal.

Solution: Direction ratios of the normal to the planes are $2, -1, 3$

$\therefore$ equation to the plane is

$2(x-1) - 1(y-2) + 3(z-3) = 0$

ie $2x - 2 - y + 2 + 3z - 9 = 0$

ie $2x - y + 3z - 9 = 0$
(2) Find the equation of the plane through the point (2, 1, 0) and perpendicular to the planes \(2x - y - z = 5\) and \(x + 2y - 3z = 5\)

Solution: Equation of the plane through the point (2, 1, 0) is \(a(x - 2) + b(y - 1) + c(z - 0) = 0\)

This plane is \(\perp\) to \(2x - y - z = 5\)
\[2a - b - c = 0 \quad (1)\]

the plane is also \(\perp\) to \(x + 2y - 3z = 5\)
\[a + 2b - 3c = 0 \quad (2)\]

Let us eliminate \(c\) between (1) & (2)

\((1)\times 3\) is \(6a - 3b - 3c = 0\)
\((2)\times 1\) is \(a + 2b - 3c = 0\)

Subtracting \(5a - 5b = 0 \Rightarrow a = b\)

from (1), \(2a - a - c = 0 \Rightarrow a = c\)
\[\therefore a = b = c\]

\(\therefore\) Equation of the plane is \(a(x - 2) + b(y - 1) + c(z - 0) = 0\)

ie \(x - 2 + y - 1 + z + 0 = 0\)

ie \(x + y + z = 3\).

(3) Find the equation of the plane through (1, 2, 3) and parallel to the plane \(4x + 5y - 3z = 7\).

Solution: Equation of the plane parallel to \(4x + 5y - 3z = 7\) can be taken as \(4x + 5y - 3z = k\) where \(k\) is a constant. The plane passes through (1, 2, 3)

\[4(1) + 5(2) - 3(3) = k \quad \text{ie} \quad 4 + 10 - 9 = k \quad \therefore k = 5\]

\(\therefore\) Equation of the required plane is \(4x + 5y - 3z = 5\).

(4) Find the \(\perp\) distance of the point (3, 2, 1) from the plane passing through the points (1, 1, 0), (3, – 1, 1) & (–1, 0, 2)

Solution: Let \(P = (3, 2, 1)\) & \(A = (1, 1, 0), B = (3, -1, 1), C = (-1, 0, 2)\)

\[\overrightarrow{AB} = 2\hat{i} - 2\hat{j} + \hat{k}\]
\[\overrightarrow{AC} = -2\hat{i} - \hat{j} + 2\hat{k}\]

\(\therefore\) \[\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{vmatrix} = \hat{i}(-4 + 1) - \hat{j}(4 + 2) + \hat{k}(2 - 4) = -3\hat{i} - 6\hat{j} - 6\hat{k}\]

\(\therefore\) direction ratios of the normal to the plane are \(-3, -6, -6\)

Equation of the plane is given by \(-3(x - 1) - 6(y - 1) - 6(z - 0) = 0\)

ie \(-3x - 6y - 6z + 3 + 6 = 0\)

ie \(x + 2y + 2z - 3 = 0\)

\(\therefore\) length of the \(\perp\) from (3, 2, 1) upon the the plane is \[\frac{3 + 4 + 2 - 3}{\sqrt{1 + 4 + 4}} = \frac{6}{3} = 2\text{ units.}\]
(5) Show that the four points \((0, -1, 0), (2, 1, -1), (1, 1, 1)\) & \((3, 3, 0)\) are coplanar. Find the equation of the plane.

Solution: Let \(A = (0, -1, 0), B = (2, 1, -1), C = (1, 1, 1)\) & \(D = (3, 3, 0)\)

Let us find the equation of the plane passing through \(A, B,\) & \(C\).

The plane passing through \((0, -1, 0)\) is given by \(a(x - 0) + b(y + 1) + c(z - 0) = 0\)

\(\text{ie } ax + by + cz + b = 0\)

\((2, 1, -1)\) lies on it

\[2a + b - c + b = 0\]

\(\text{ie } 2a + 2b - c = 0\) \hspace{1cm} (1)

\((1, 1, 1)\) lies on it

\[a + 2b + c = 0\] \hspace{1cm} (2)

from (1) & (2) \[\frac{a}{2} + \frac{b}{2} = \frac{b}{1 - 2} = \frac{c}{4 - 2}\]

\[\text{ie } a = \frac{-b}{3} = \frac{c}{2}\]

\[\therefore \text{ Equation of the plane is } 4(x - 0) - 3(y + 1) + 2(z - 0) = 0\]

\[\text{ie } 4x - 3y + 2z - 3 = 0.\]

Consider \(D = (3, 3, 0)\)

it can be seen that it lies on the plane. \(\therefore\) the given points are coplanar.

(6) Find the distance between the parallel planes \(2x - 2y + z + 3 = 0\) & \(4x - 4y + 2z + 5 = 0\).

Solution: Now \((0, 0, -3)\) is a point on \(2x - 2y + z + 3 = 0\).

\(\perp\) distance from \((0, 0, -3)\) upon the plane \(4x - 4y + 2z + 5 = 0\)

is \[\frac{4(0) - 4(0) + 2(-3) + 5}{\sqrt{16 + 16 + 4}}\]

\[= \frac{-6 + 5}{6} = \frac{-1}{6} = \frac{1}{6}\]

\[\therefore \text{ distance between parallel planes is } \frac{1}{6}\]

(7) Find the angle between the planes \(x + 2y - 3z - 2 = 0\) and \(2x + y + z + 3 = 0\).

Solution: If \(\theta\) is the angle between two planes, we have

\[\cos \theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}\]

\[= \frac{2 + 2 - 3}{\sqrt{1 + 4 + 9} \sqrt{14} + 1} = \frac{1}{\sqrt{7} \times 2} = \frac{1}{2\sqrt{21}}\]

\[\therefore \theta = \cos^{-1}\left(\frac{1}{2\sqrt{21}}\right)\]
(8) Find the equation of the plane passing through the line of intersection of the planes \( x + y + z = 1 \) and \( 2x + 3y - z + 4 = 0 \) and perpendicular to the plane \( 2y - 3z = 4 \)

Solution: Equation of the plane passing through the line of intersection of the given two planes can be taken as

\[
(x + y + z - 1) + \lambda(2x + 3y - z + 4) = 0
\]

where \( \lambda \) is a constant.

\[
\text{ie } (1 + 2\lambda)x + (1 + 3\lambda)y + (1 - \lambda)z - 1 + 4\lambda = 0
\]

this plane is \( \perp \) to \( 2y - 3z = 4 \)

\[
\therefore 0(1 + 2\lambda) + 2(1 + 3\lambda) - 3(1 - \lambda) = 0
\]

\[
\text{ie } 2 + 6\lambda - 3 + 3\lambda = 0
\]

\[
\text{ie } 9\lambda = 1 \Rightarrow \lambda = \frac{1}{9}
\]

\[
\therefore \text{Equation of plane is } (x + y + z + 1) + \frac{1}{9}(2x + 3y - z + 4) = 0
\]

\[
\text{ie } 9x + 9y + 9z - 9 + 2x + 3y - z + 4 = 0
\]

\[
\text{ie } 11x + 12y + 8z - 5 = 0
\]

is the equation of required plane.

The Straight Line

1. Two planes intersect along a straight line, therefore \( a_1x + b_1y + c_1z + d_1 = 0 \) and \( a_2x + b_2y + c_2z + d_2 = 0 \) taken together represent a line. These are called General equations of the line.

2. If a line passes through the point \( A(x_1, y_1, z_1) \) & have \( l, m, n \) as direction cosines then

Let \( P(x, y, z) \) be any point on the line

\[
\overrightarrow{AP} = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}
\]

and direction cosines are \( l, m, n \)

\[
\therefore \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}
\]

represent two equations of the line. This is called Symmetric form of the equation of the line.

3. If \( a, b, c \) are the direction ratios of the line passing through the point \( (x_1, y_1, z_1) \), then equations of the line are

\[
\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.
\]

If the line passes through the points \( (x_1, y_1, z_1) \) & \( (x_2, y_2, z_2) \), then the direction ratios of the line are \( x_2 - x_1, y_2 - y_1, z_2 - z_1 \)

\[
\therefore \text{its equations are } \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}
\]
4. To find the angle between the line \( \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \) and the plane \( ax+by+cz+d=0 \).

Solution: If \( \theta \) is the normal angle between the line and the plane

\[ \cos(90^\circ - \theta) = \frac{la+mb+nc}{\sqrt{l^2+m^2+n^2} \sqrt{a^2+b^2+c^2}} \]

\[ \sin \theta = \frac{la+mb+nc}{\sqrt{a^2+b^2+c^2}}. \]

If the line is parallel to the plane then \( \theta = 0^\circ \)

\[ \therefore \text{line is parallel to the plane.} \]

If the line is perpendicular to the plane then \( \frac{l}{a} = \frac{m}{b} = \frac{n}{c} \)

Examples

(1) Show that the line \( \frac{x-1}{3} = \frac{y+2}{-2} = \frac{z-1}{2} \) is parallel to the plane \( 2x+2y-z=6 \) and find the distance between them.

Solution: Direction ratios of the normal to the plane \( 2x+2y-z=6 \) are \( 2, 2, -1 \)

Direction ratios of the given line are \( 3, -2, 2 \)

Now \( 2(3) + 2(-2) - 1(2) = 6 - 4 - 2 = 0. \)

\[ \therefore \text{line is perpendicular to the normal to the plane.} \]

\( (1, -2, 1) \) is a point on the given line. \( \therefore \text{distance of \((1, -2, 1)\) upon the plane } 2x+2y-z=6 \)

\[ \text{is } \frac{|2(1)+2(-2)-1-6|}{\sqrt{4+4+1}} = \frac{|2-4-7|}{3} = \frac{9}{3} = 3. \]

(2) Find the equation of the line through \((1, 2, -1)\) perpendicular to each of the lines

\[ \frac{x}{1} = \frac{y}{0} = \frac{z}{-1} \]

and

\[ \frac{x}{3} = \frac{y}{4} = \frac{z}{5}. \]

Solution: Let \( a, b, c \) be the direction ratios of the required line.

Then \( a+o-c=0 \) & \( 3a+4b+5c=0. \)

\[ \therefore \frac{a}{0+4} = \frac{b}{-3-5} = \frac{c}{4-0} \]

\[ \therefore \frac{a}{4} = \frac{b}{-8} = \frac{c}{4} \]

\[ \therefore \frac{a}{1} = \frac{b}{-2} = \frac{c}{1} \]

\[ \therefore \text{equations of the required line are } \frac{x-1}{1} = \frac{y-2}{-2} = \frac{z+1}{3}. \]

(3) Find the angle between the line \( \frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6} \) and the plane \( 3x+y+z=7. \)

Solution: If \( \theta \) is the angle between line and the plane, then
\[
\sin \theta = \frac{(2)(3) + 3(1) + 6(1)}{\sqrt{4 + 9 + 36} \sqrt{9 + 1 + 1}} = \frac{6 + 3 + 6}{7\sqrt{11}} = \frac{15}{7\sqrt{11}}
\]

\[\therefore \theta = \sin^{-1} \left( \frac{15}{7\sqrt{11}} \right)\]

(4) Find the perpendicular distance of the point \((1, 1, 1)\) from the line \(\frac{x-2}{2} = \frac{y+3}{2} = \frac{z}{-1}\)

Solution: Let \(A(2, -3, 0)\) be a point on the line & \(P(1, 1, 1)\) be the given point

\[\therefore AP = \sqrt{1 + 16 + 1} = \sqrt{18} = 3\sqrt{2}\]

\[AP = O\bar{P} = \hat{O}A - \hat{O}P = \frac{-\hat{i} + 4\hat{j} + \hat{k}}{\sqrt{4 + 4 + 1}}\]

Projection of \(AP\) on the line \(AQ = (-\hat{i} + 4\hat{j} + \hat{k}) \cdot \frac{(2\hat{i} + 2\hat{j} - \hat{k})}{\sqrt{4 + 4 + 1}}\)

\[= -2 + 8 - 1 = \frac{5}{3}\]

\[\therefore PQ = AP - AQ = 18 - \frac{25}{9} = \frac{162 - 25}{9} = \frac{137}{9}\]

\[\therefore PQ = \frac{\sqrt{137}}{3}\]

Alternate Method

Let \(\frac{x - 2}{2} = \frac{y + 3}{2} = \frac{z}{-1} = r\) any point on the line is \((2r + 2, 2r - 3, -r)\)

direction ratios of the line joining this point & the point \((1, 1, 1)\) are \(2r + 1, 2r - 4, -r - 1\)

This line will be \(\perp\) to the given line if \(2(2r + 1) + 2(2r - 4) - 1(-r - 1) = 0\)

ie \(4r + 2 + 4r - 8 + r + 1 = 0\)

ie \(9r - 5 = 0\) ie \(r = \frac{5}{9}\)

\[\therefore \text{Co-ordinates of the foot of the perpendicular are } \left( \frac{28}{9}, \frac{-17}{9}, \frac{-5}{9} \right) \text{ ie } \left( \frac{28}{9}, \frac{-17}{9}, \frac{-5}{9} \right)\]

Distance between \(\left( \frac{28}{9}, \frac{-17}{9}, \frac{-5}{9} \right) \) & \((1, 1, 1)\) is \(\sqrt{\left( \frac{28}{9} - 1 \right)^2 + \left( \frac{-17}{9} - 1 \right)^2 + \left( \frac{-5}{9} - 1 \right)^2}\)

\[= \sqrt{\frac{19^2 + 26^2 + 14^2}{9}} = \frac{\sqrt{9 \times 137}}{9} = \frac{\sqrt{137}}{3}\]

(5) Find the image of the point \((1, 3, 4)\) in the plane \(2x - y + z + 3 = 0\)

Solution: Let \(P\) be the given point and \(Q\) be its image on the plane \(2x - y + z + 3 = 0\)

Equations to \(PQ\) are \(\frac{x - 1}{2} = \frac{y - 3}{-1} = \frac{z - 4}{1} = r\) (say)
Then co-ordinates of $Q$ are $(2r+1, -r+3, r+4)$
Mid-point of $PQ$ is given by
\[
\left( \frac{2r+1+1}{2}, \frac{-r+3+3}{2}, \frac{r+4+4}{2} \right)
\]
\[
\left( r+1, \frac{-r+6}{2}, \frac{r+8}{2} \right)
\]
this point lie on the plane.
\[
2(r+1) - \left( \frac{-r+6}{2} \right) + \frac{r+8}{2} + 3 = 0
\]
\[
4r + 4 + r - 6 + r + 8 + 6 = 0 \quad \text{ie} \quad 6r = -12 \Rightarrow r = -2
\]
\[
Q = (-4+1, 2+3, -2+4) = (-3, 5, 2)
\]
\[
\text{Image of } (1, 3, 4) \text{ in the given plane is } (-3, 5, 2)
\]

(6) To find the condition for two lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ to intersect (or to be coplanar) and to find the equation of the plane.

Solution: The equation of the plane containing the first line is given by $a(x-x_1)+b(y-y_1)+c(z-z_1) = 0 \quad (1)$
where $al_1 + bm_1 + cn_1 = 0 \quad (2)$
the second line will lie in the plane if $al_2 + bm_2 + cn_2 = 0 \quad (3)$
and $(x_2, y_2, z_2)$ satisfy (1)
\[
\text{ie} \quad a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1) = 0 \quad (4)
\]
elminating $a, b, c$ from (2), (3) & (4)

We get
\[
\begin{vmatrix}
\frac{x-x_1}{l_1} & \frac{y-y_1}{m_1} & \frac{z-z_1}{n_1} \\
\frac{x-x_2}{l_2} & \frac{y-y_2}{m_2} & \frac{z-z_2}{n_2}
\end{vmatrix} = 0
\]
which is the required condition for coplanarity of two lines.

Equation to the plane is by eliminating $a, b, c$ from (1), (2) & (3)

\[
\begin{vmatrix}
\frac{x-x_1}{l_1} & \frac{y-y_1}{m_1} & \frac{z-z_1}{n_1} \\
\frac{x-x_2}{l_2} & \frac{y-y_2}{m_2} & \frac{z-z_2}{n_2}
\end{vmatrix} = 0
\]
is the required equation of the plane.

**Exercise**
1. Find the equation of the plane passing through the point $(-2, 2, 2)$ and containing the line joining the points $(1, 1, 1)$ & $(1, -1, 2)$.
   (Ans: $x - 3y - 6z + 8 = 0$)
2. Show that the points $(-6, 3, 2), (3, -2, 4), (5, 7, 3)$ and $(-13, 17, -1)$ are coplanar.
3. Find the equation of the plane through the points $(2, 2, 1)$ and $(9, 3, 6)$ and perpendicular to the plane $2x + 6y + 6z = 9$.
   (Ans: $3x + 4y - 5z = 0$)
4. Find the equation of the plane through the points whose position vectors are $3\hat{i} - \hat{j} + \hat{k}$, $2\hat{j} - \hat{k}$ and $\hat{i} + \hat{j} + \hat{k}$.
   (Ans: $2x + 2y + z = 5$)

5. Find the equation of the plane through the points $(2, 2, 1), (1, -2, 3)$ and parallel to the joining the points $(2, 1, -3), (-1, 5, -8)$.
   (Ans: $12x - 11y - 16z + 14 = 0$)

6. Find the equation of the plane which contains the line $\frac{x-1}{2} = \frac{y-3}{1} = \frac{z-13}{2}$ and is perpendicular to the plane $x + y + z = 3$.
   (Ans: $x - z + 12 = 0$)

7. Find the equation of the plane through the line $\frac{x-1}{3} = \frac{y-4}{2} = \frac{z-4}{-2}$ and parallel to the line $\frac{x+1}{3} = \frac{y-1}{-4} = \frac{z+2}{1}$.
   (Ans: $2x + 3y + 6z = 38$)

8. Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar and find the equation of the plane containing them.
   (Ans: $x - 2y + z = 0$)

9. Show that the lines $\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3}$ and $\frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}$ are coplanar and find the equation of the plane containing them.
   (Ans: $6x - 5y - z = 0$)

10. Show that the line $x + 10 = \frac{8 - y}{2} = z$ lies in the plane $x + 2y + 3z = 6$.

11. Find the co-ordinates of the foot of the perpendicular drawn from the point $(-1, -3, 2)$ upon the plane $3x + 4y + 5z = 5$.
    
    \[ \text{Ans: } \left(-\frac{2}{5}, -\frac{11}{5}, 3\right) \]

12. Prove that the lines $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ and $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ intersect and find the co-ordinates of the point of intersection.
    
    \[ \text{Ans: } (5, -7, 6) \]

**Sphere**

**Definition :-** A Sphere is the locus of a point which remains at a constant distance from a fixed point in three dimension.

The fixed point is the centre and constant distance is called **radius**.

Equation of the sphere whose centre is at $(x_1, y_1, z_1)$ and radius $r$ is given by

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$$

1. To show that $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere.
   
   The expression can be written as
   $$(x^2 + 2ux) + (y^2 + 2vy) + (z^2 + 2wz) + d = 0$$
   
   ie $$(x + u)^2 - u^2 + (y + v)^2 - v^2 + (z + w)^2 - w^2 + d = 0$$
ie \((x+u)^2 + (y+v)^2 + (z+w)^2 = u^2 + v^2 + w^2 - d\)

which represents a sphere whose centre is \((-u, -v, -w)\) and radius is \(\sqrt{u^2 + v^2 + w^2 - d}\)

2. **Plane section of a sphere**

Plane section of a sphere is always a circle. If the plane passes through the centre of the sphere it is called a great circle, otherwise a small circle.

Let \(C\) be the centre of the sphere and \(A\) be a point on the sphere and on the plane which intersects the sphere. Let \(B\) be the centre of the small circle then \(BC\) is \(\perp\) to \(AB\).

\[
\therefore AC^2 = AB^2 + BC^2
\]

\(BC = p\) & \(AC = R\) (radius of the sphere)

Then \(AB^2 = AC^2 - BC^2 = R^2 - p^2\)

\[
\therefore \text{radius of the small circle is } AB = \sqrt{R^2 - p^2}
\]

Let \(S: x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0\) be the equation of the sphere

and \(P: ax + by + cz + d = 0\) be the plane.

Then the equation of the sphere passing through the circle is \(S + \lambda P = 0\) where \(\lambda\) is a constant.

If \(S_1 = 0\) ie \(x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0\) & \(S_2 = 0\) ie \(x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0\) represents two spheres then the equation of the circle common to \(S_1 = 0\) & \(S_2 = 0\) is \(S_1 - S_2 = 0\)

\[
\therefore \text{The equation of the sphere through the circle is given by } S_1 + \lambda(S_1 - S_2) = 0.
\]

3. To find the equation of the sphere having the \((x_1, y_1, z_1)\) & \((x_2, y_2, z_2)\) as the extremities of a diameter.

Let \(A\) & \(B\) be the extremities of the diameter.

Let \(P(x, y, z)\) be a point on the sphere.

\(AP\) is \(\perp\) to \(PB\).

Direction ratios of \(AP\) are \(x-x_1, y-y_1, z-z_1\)

Direction ratios of \(BP\) are \(x-x_2, y-y_2, z-z_2\)

Since \(AP\) is \(\perp\) to \(PB\),

\((x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0\).

4. To find the equation of the tangent plane at \((x_1, y_1, z_1)\) on the sphere \(x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0\)

Let \(Q(x, y, z)\) be any point in the tangent plane then direction ratios of \(PQ\) are \(x-x_1, y-y_1, z-z_1\) and direction ratios of \(PC\) are \(x_1+u, y_1+v, z_1+w\) where \(C\) is the centre of the sphere.
As $PQ$ is $\perp$ to $PC$, we have
\[(x-x_1)(x_1+u)+(y-y_1)(y_1+v)+(z-z_1)(z_1+w)=0\]
ie $xx_1-x_1^2+ux-ux_1+yy_1-y_1^2+vy-yy_1$
$+zz_1-z_1^2+wz-wz_1=0$

ie $xx_1+yy_1+zz_1+ux-ux_1+vy-yy_1+wz-wz_1$
$-x_1^2-y_1^2-z_1^2=0$

Since $(x_1, y_1, z_1)$ is a point on the sphere
\[x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0\]
∴ $-x_1^2-y_1^2-z_1^2=2ux_1+2vy_1+2wz_1+d$

substituting this in (1) and simplifying, we have
\[xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0\]
is the equation to the tangent plane.

5. To find the condition for two spheres to cut orthogonally (ie. the tangents plane at a point of intersection are at right angles).

Let $C_1$ & $C_2$ be the centres of two spheres whose radii are $r_1$ & $r_2$, $P$ is point of intersection.

Let the spheres be
\[S_1: x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0\]
\[S_2: x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0\]

Now $C_1P=C_2=90^\circ \because C_1C_2=r_1^2+r_2^2$

ie $(-u_1+u_2)^2+(-v_1+v_2)^2+(-w_1+w_2)^2$
\[=\left(\sqrt{u_1^2+v_1^2+w_1^2-d_1}\right)^2+\left(\sqrt{u_2^2+v_2^2+w_2^2-d_2}\right)^2\]
ie $u_1^2+w_1^2-2u_1u_2+v_1^2+v_2^2-2v_1v_2+w_1^2+w_2^2-2w_1w_2=u_1^2+v_1^2+w_1^2-d_1+u_2^2+v_2^2+w_2^2-d_2$
\[-2u_1u_2-2v_1v_2-2w_1w_2=-d_1-d_2\]
ie $2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$
is the required condition.

Note: If $ax^2+ay^2+az^2+2ux+2vy+2wz+d=0$ be the given equation then it can be reduced to the standard form as
\[x^2+y^2+z^2+\frac{2u}{a}x+\frac{2v}{a}y+\frac{2w}{a}z+\frac{d}{a}=0\]

Examples

1. Find the equation of the sphere whose centre is at $(2, -3, 4)$ and radius 3 units.

Solution: The required equation is $(x-2)^2+(y+3)^2+(z-4)^2=3^2$
ie $x^2-4x+4+y^2+6y+9+z^2-8y+16-9=0$
\[\text{ie } x^2+y^2+z^2-4x+6y-8z+20=0\]
2. Find the centre and radius of the sphere \(x^2 + y^2 + z^2 - 6x + 4y - 3z - \frac{3}{4} = 0\)

Solution: Comparing with standard equation \(2u = -6, \ 2v = 4, \ 2w = -3, \ d = -\frac{3}{4}\)

\[
\therefore u = -3, \ v = 2, \ w = -\frac{3}{2}
\]

\[
\therefore \text{Centre} = (-u, -v, -w) = \left(3, -2, \frac{3}{2}\right)
\]

\[
\text{radius } r = \sqrt{9 + 4 + \frac{9}{4} + \frac{3}{4}} = \sqrt{13 + \frac{9 + 3}{4}} = \sqrt{16} = 4.
\]

3. Find the equation of the sphere whose diameter is the line joining the points \((4, 0, -2)\) and \((0, 3, 1)\).

Solution: Required equation is of the form \((x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0\)

\[
\therefore \text{Required equation is } (x-4)(x-0) + (y-0)(y-3) + (z+2)(z-1) = 0
\]

\[
\text{ie } x^2 - 4x + y^2 - 3y + z^2 + z - 2 = 0
\]

\[
\text{ie } x^2 + y^2 + z^2 - 4x - 3y + z - 2 = 0
\]

4. Find the equation of the sphere through the circle \(x^2 + y^2 + z^2 = 9, \ 2x + 3y + 4z = 5\) and the point \((1, 2, 3)\).

Solution: The required equation is \(x^2 + y^2 + z^2 - 9 + \lambda(2x + 3y + 4z - 5) = 0\), \((1, 2, 3)\) lie on it

\[
\therefore 1 + 4 + 9 - 9 + \lambda(2 + 6 + 12 - 5) = 0
\]

\[
\text{ie } 5 + \lambda(15) = 0 \Rightarrow \lambda = -\frac{1}{3}
\]

\[
\therefore \text{Equation of the sphere is } (x^2 + y^2 + z^2 - 9) - \frac{1}{3}(2x + 3y + 4z - 5)
\]

\[
\text{ie } 3x^2 + 3y^2 + 3z^2 - 27 - 2x - 3y - 4z + 5 = 0
\]

\[
\text{ie } 3x^2 + 3y^2 + 3z^2 - 2x - 3y - 4z - 22 = 0
\]

5. Find the equation of the sphere for which the circle \(x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 = 0, \ 5x - 2y + 4z + 7 = 0\) is a great circle.

Solution: Required equation of the sphere is \(x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 + \lambda(5x - 2y + 4z + 7) = 0\)

Centre of this sphere is \(\left[\frac{-1}{2}(-3 + 5\lambda), \frac{-1}{2}(4 - 2\lambda), \frac{-1}{2}(-2 + 4\lambda)\right]\)

Circle will be a great circle if the centre lies on the plane.

\[
\therefore -\frac{5}{2}(-3 + 5\lambda) + 4 - 2\lambda) - 2(-2 + 4\lambda) + 7 = 0
\]

\[
\text{ie } \frac{15}{2} - \frac{25}{2} \lambda + 4 - 2\lambda + 4 = 8\lambda + 7 = 0
\]

\[
\text{ie } -\frac{45}{2} \lambda + \frac{45}{2} = 0 \Rightarrow \lambda = 1
\]
6. Find the equation of the tangent plane to the sphere \(3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0\) at the point \((1, 2, 3)\)

Solution: Equation of the sphere is \(x^2 + y^2 + z^2 = \frac{2}{3}x - y - \frac{4}{3}z - \frac{22}{3} = 0\)

Tangent plane at \((1, 2, 3)\) is \(x(1) + y(2) + z(3) - \frac{1}{3}(x + 1) - \frac{1}{2}(y + 2) - \frac{2}{3}(z + 3) - \frac{22}{3} = 0\)

\[\text{ie } x + 2y + 3z - \frac{1}{3}x - \frac{1}{3}y - 1 - \frac{2}{3}z - 2 - \frac{22}{3} = 0\]

Multiply through out by 6

\[6x + 12y + 18z - 2x - 2 - 3y - 6 - 4z - 12 - 44 = 0\]

\[\text{ie } 4x + 9y + 14z - 64 = 0\]

7. Show that the spheres \(x^2 + y^2 + z^2 + 6y + 14z + 8 = 0\) and \(x^2 + y^2 + z^2 + 6x + 4z + 20 = 0\) intersect at right angles.

Solution: Condition for two spheres to be orthogonal is \(2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2\)

For the first sphere \(u_1 = 0, v_1 = 3, w_1 = 7, d_1 = 8\)

For the second sphere \(u_2 = 3, v_2 = 0, w_2 = 2, d_2 = 20\)

\[d_1 + d_2 = 8 + 20 = 28\]

\[\therefore 2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2\]

\[\therefore \text{The two spheres intersect orthogonally.}\]

8. Show that the plane \(2x - 2y + z + 12 = 0\) touches the sphere \(x^2 + y^2 + z^2 - 2x - 4y + 2z = 3\) and find the point of contact.

Solution: Let C be the centre of the sphere, then C = \((1, 2, -1)\).

Length of the \(\perp\) from C upon the plane

\[2x - 2y + z + 12 = 0 \quad \text{is} \quad \frac{2 - 4 - 1 + 12}{\sqrt{4 + 4 + 1}} = \frac{3}{3} = 3,\]

Let P be the point of Contact then CP is \(\perp\) to the plane.

\[\therefore \text{Equation to } CP \text{ can be taken as } \frac{x - 1}{3} = \frac{y - 2}{3} = \frac{z + 1}{3}, \text{ say}\]

\[\therefore \text{Co - ordinates of } P \text{ can be taken as } (2r + 1, -2r + 2, r - 1)\]

This will lie in the plane if \(2(2r + 1) - 2(-2r + 2)(r - 1) + 12 = 0\)

\[\Rightarrow 4r + 2 + 4r - 4 + r - 1 + 12 = 0\]

\[9r + 9 = 0 \Rightarrow r = -1\]

\[\therefore \text{Co - ordinates of } P \text{ are } (-2 + 1, 2 + 2, -1 - 1) = (-1, 4, -2)\]

\[\therefore \text{Point of contact is } (-1, 4, -2)\]
Right Circular Cone

Definition: A right circular cone is a surface generated by a straight line which passes through a fixed point (called Vertex) and makes a constant angle with a fixed line (called axis).

The constant angle is called semi-vertical angle of the cone.

Examples

(1) Find the equation of the right circular cone whose vertex is at the origin, semi-vertical angle is $\alpha$ and having axis of $Z$ as its axis.

Solution: Let $P(x, y, z)$ be any point on the generator. $OZ$ is the axis whose direction cosines are 0, 0, 1.

Direction ratios of $OP$ are $x$, $y$, $z$

$\therefore \cos \alpha = \frac{x \cdot 0 + y \cdot 0 + z \cdot 1}{\sqrt{x^2 + y^2 + z^2} \cdot 1} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$

Squaring both sides

$\cos^2 \alpha = \frac{z^2}{x^2 + y^2 + z^2}$

$\therefore x^2 + y^2 + z^2 = z^2 \sec^2 \alpha$

ie $x^2 + y^2 = z^2 (\sec^2 \alpha - 1) = z^2 \tan^2 \alpha$

$\therefore$ Equation of the cone is $x^2 + y^2 = z^2 \tan^2 \alpha$

(2) Find the equation of the right circular cone with semi-vertical angle $30^\circ$, vertex at the point $(2, 1, -3)$ and the direction ratios of whose axis are 3, 4, $-1$.

Solution: Let $P(x, y, z)$ be any point on the generator and vertex $V = (2, 1, -3)$

Direction ratios of $PV$ are $x-2$, $y-1$, $z+3$

Direction ratios of axis are 3, 4, $-2$ & $\alpha = 30^\circ$

$\therefore \cos 30^\circ = \frac{3(x-2) + 4(y-1) - (z+3)}{\sqrt{(x-2)^2 + (y-1)^2 + (z+3)^2}} \frac{\sqrt{9+16+1}}{2}$

ie $\frac{\sqrt{3}}{2} = \frac{3x+4y-z-6-4-3}{\sqrt{(x-2)^2 + (y-1)^2 + (z+3)^2}} \frac{\sqrt{26}}{2}$